Université Paris Dauphine

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Introduction to Time series

TD3 - ARMA Process

Exercice 1

1. Solve the following ARMA equation:
$$X_t = 3X_{t-1} + Z_t - \frac{10}{3}Z_{t-1} + Z_{t-2} \qquad t \in \mathbb{Z},$$
where Z is white noise with zero mean and variance σ^2. Specify the mean and the auto-covariance function of the solution.

2. Same question for
$$X_t = X_{t-1} - \frac{1}{4}X_{t-2} + Z_t + Z_{t-1} \qquad t \in \mathbb{Z}.$$

Exercice 2 (ARMA$(1,1)$) Consider the equation
$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$$
where (Z_t) is white noise with zero mean and variance σ^2 and $\phi, \theta \in \mathbb{R}$

1. If $\phi \neq \pm 1$, show that there exists a unique stationary solution; calculate it and find its mean and its auto-covariance function.
2. if $\phi = 1$ and X is the solution, show that, for all $t \geq 1$,
$$X_t = X_0 + \theta Z_0 + (1+\theta)\sum_{s=1}^{t-1} Z_s + Z_t.$$
Deduce that, if $\theta \neq -1$, there is no stationary solution.
3. Show similarly that there is no stationary solution if $\phi = -1$ and $\theta \neq 1$.
4. We now assume that $\phi = 1$ and $\theta = -1$. Show that the solutions of the equation are processes of the form $X_t = Z_t + \xi$, where ξ is a random variable. Show that such a process is stationary, if and only if, ξ is square integrable and uncorrelated from Z.
5. Find in the same way the stationary solutions when $\phi = -1$ and $\theta = 1$.

Exercice 3 (MA(1)) Suppose (Z_t) is white noise with zero mean and variance σ^2, θ is a real number and X is a process given by $X_t = Z_t + \theta Z_{t-1}$ for $t \in \mathbb{Z}$. Show that X is a stationary process and its autocovariance function is
$$\gamma_X(h) = \begin{cases} a & \text{si } h = 0 \\ b & \text{si } h = \pm 1 \\ 0 & \text{sinon} \end{cases}$$
where a and b are real numbers that will be determined, with $|b| \leq a/2$.

Now we want to show that, if X is a stationary process possessing autocovariance function of this form, then there is a white noise Z and a real number θ such that $X_t = Z_t + \theta Z_{t-1}$ for all $t \in \mathbb{Z}$.

1. For now we suppose that $|b| < a/2$. Show that we can solve the problem by choosing $|\theta| < 1$ and Z a causal filter of X. But if $|b| = a/2$, what happens ?

2. Now we suppose $b = -a/2$. We set $Y_n = \sum_{k=1}^{n} X_k$ for all $n \geq 1$.

   (a) Show that $\text{Var}(Y_n) = a$ and $\text{Cov}(Y_n, Y_m) = a/2$ for all different integers n and m.

   (b) Deduce that $\frac{1}{n}\sum_{k=1}^{n} Y_k$ converges to a random variable U in L^2.

   (c) Show that the random variables $Y_n - U$ are uncorrelated and then conclude.

3. Similarly solve the problem when $b = a/2$.

Exercice 4 Let Z and W be uncorrelated standard white noises and let $\phi, \psi \in [0, 1)$. Consider the stationary solutions of X and Y of

$$\begin{cases} Y_t = \phi Y_{t-1} + X_t + W_t \\ X_t = \psi X_{t-1} + Z_t \end{cases}$$

1. Show that the second equation defines a unique stationary solution X. Express the solution and calculate the mean and the autocovariance function.

2. Show that $X + W$ is stationary, calculate its mean and autocovariance function.

3. Deduce that the first equation defines a unique stationary process Y.

4. For all $t \in \mathbb{Z}$, set $V_t = Y_t - (\phi + \psi)Y_{t-1} + \phi\psi Y_{t-2}$. Using Exercise 3, show that there exists a real number θ and a white noise H such that $V_t = H_t + \theta H_{t-1}$ for all $t \in \mathbb{Z}$.

5. Deduce that Y satisfies some ARMA equation driven by the white noise H.

6. Solve this ARMA equation (we can distinguish the two cases $\phi = \psi$ and $\phi \neq \psi$).

Exercice 5 (Lack of uniqueness of the ARMA solution) Let P be a polynomial with real coefficients and P has a root of modulus 1.

1. We denote by B the lag operator. Show that there exists a harmonic process Y solution of $P(B)Y = 0$ (i.e., there exist $\theta \in \mathbb{R}$, U and V of centered random variables with variance 1 and uncorrelated, such that $Y = U\cos(\theta t) + V\sin(\theta t)$ satisfies $P(B)Y = 0$).

2. Let Q be a polynomial whose roots of modulus 1 compensate those of P (i.e. all the roots of P of modulus 1 are also roots of Q). Show that the ARMA equation $P(B)X = Q(B)Z$ has a stationary solution which is not a filter of Z.